

# A NOTE ON THE GAUSS-BONNET-CHERN THEOREM FOR GENERAL CONNECTION

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**ABSTRACT.** In this paper, we prove a local index theorem for the DeRham Hodge-Laplacian which is defined by the connection compatible with metric. This connection need not be the Levi-Civita connection. When the connection is Levi-Civita connection, this is the classical local Gauss-Bonnet-Chern theorem.

## 1. INTRODUCTION

The Gauss-Bonnet-Chern theorem is an index theorem about relationship between topology and geometry on a compact manifold. It has been proved by Allendoerfer-Weil [7] in 1940s. Later, Chern [8] has given a proof by intrinsic computation. The refined local Gauss-Bonnet-Chern theorem was proved by Patodi [9] in 1971, which was conjectured by McKean-Singer [10]. In the above theorem, the DeRham operator and Hodge-Laplacian are defined by Levi-Civita connection. Recently, Beneventano-Gilkey-Kirsten-Santangelo [4] have studied the Gauss-Bonnet theorem for general connection and corresponding heat trace's asymptotic expansion. Bell [5] has given Gauss-Bonnet theorem for vector bundle whose rank is equal to the dimension of underlying manifold. There have been also some works about Gauss-Bonnet-Chern theorem's generalization in Finsler geometry (see Bao-Chern-Shen [15], Lackey [16], Zhao [17] and so on).

Now we state the main theorem in this article.

**Theorem 1.1.** *Let  $M$  be a compact Riemannian manifold of even dimension  $d$  ( $d = 2l$ ) with metric  $g$ , which has a metric compatible connection  $D$ . Let  $\varepsilon(e)$  denote exterior multiplication by differential form  $e$  and  $\iota(e)$  denote interior multiplication by  $e$ . Then we may define DeRham Dirac operator, for exterior form bundle section  $f$ :*

$$\mathcal{D}f = (\varepsilon(e^i) - \iota(e^i))D_{e_i}f$$

Where  $\{e_i\}$  is tangent vector frame,  $\{e^i\}$  is its dual frame. Let  $h(t, x, y)$  be the heat kernel (fundamental solution) for the following DeRham Hodge-laplacian equation:

$$\frac{\partial}{\partial t}f = -\frac{1}{2}\mathcal{D}^2f$$

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Then we have

$$\lim_{t \rightarrow 0} \text{Str}\{h(t, x, x)\} dm = \frac{1}{(2\pi)^l} Pf(-R), \quad (1.1)$$

where  $\text{Str}$  denote supertrace,  $dm$  is the volume element,  $R$  is the Riemannian curvature corresponding to  $D$ ,  $Pf$  is Pfaffian.

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## 2. GEOMETRIC PRELIMINARES

Let  $\widehat{D}$  denote Levi-Civita connection,  $\{e_i\}$  be orthonormal tangent frame.  $\widetilde{D}(e_i) = \widehat{\Gamma}_{li}^s e^l \otimes e_s$ ,  $(D)(e_i) = \Gamma_{li}^s e^l \otimes e_s$ . Let

$$\wedge T_x^* M = \wedge^+ T_x^* M \oplus \wedge^- T_x^* M,$$

for  $x \in M$ , where  $\wedge^+ T_x^* M$  consists of even degree forms,  $\wedge^- T_x^* M$  consists of odd degree forms. For  $a \in \text{End}(\wedge T_x^* M)$ , define

$$\text{Str}(a) = \text{trace}(a_{\wedge^+}) - \text{trace}(a_{\wedge^-}).$$

The fundamental solution's asymptotic expansion in  $(x, x)$  is determined by local condition around  $x$  (see [4] chapter 2). So we may assume  $M$  is spin manifold, whose spinor bundle is denoted by  $\mathcal{S}$ , dual bundle is denoted by  $\mathcal{S}^*$ . For connection  $\widehat{D}$  and  $D$  respectively,  $\mathcal{S}$  and  $\mathcal{S}^*$  have lifted connections, which are denoted by  $\widehat{D}^{\mathcal{S}}$  and  $\widehat{D}^{\mathcal{S}^*}$ ,  $D^{\mathcal{S}}$  and  $D^{\mathcal{S}^*}$ . There is a linear isomorphism between Clifford bundle  $Cl(T^*M)$  and exterior form bundle  $\Omega^*(M)$ , which is not an algebraic isomorphism:

$$Cl(T^*M) \cong \Omega^*(M). \quad (1.2)$$

The isomorphism (1.2) is denoted by  $c$ , the inverse of  $c$  by  $\sigma$ .

We still denote complexified Clifford algebra bundle by  $Cl(T^*M)$ .  $Cl(T^*M)$  has an action on  $\mathcal{S}$ . The lifted connection on  $\mathcal{S}$  is compatible with this Clifford action. There is also an isomorphism between the complexified Clifford algebra bundle  $Cl(T^*M)$  and the endmorphism bundle  $\mathcal{S} \otimes \mathcal{S}^*$  of  $\mathcal{S}$ :

$$Cl(T^*M) \cong \mathcal{S} \otimes \mathcal{S}^*. \quad (1.3)$$

This isomorphism is an algebraic isomorphism. For a orthonormal base  $\{e^i\} \in T_x^*M$ , we define a chirality element:

$$\Gamma = i^l c(e^1)c(e^2)\dots c(e^d).$$

By computations, one could get  $\Gamma^2 = 1$ . Let

$$\mathcal{S}^+ = \{a | \Gamma a = a, a \in \mathcal{S}_x\},$$

$$\mathcal{S}^- = \{a | \Gamma a = -a, a \in \mathcal{S}_x\}.$$

By isomorphism (1.2) and (1.3), there is:

$$\Omega^*(M) \cong \mathcal{S} \otimes \mathcal{S}^*. \quad (1.4)$$

As exterior form bundle,  $\Omega^*(M)$  has a connection which is from  $T^*M$ 's connection, a graded structure in terms of even and odd degree.  $Cl(T^*M)$  has an action on it. As tensor product of  $\mathcal{S}$  and  $\mathcal{S}^*$ ,  $\mathcal{S} \otimes \mathcal{S}^*$  has an connection from  $\mathcal{S}$ , a graded structure

$$((\mathcal{S}^+ \otimes \mathcal{S}^{*+}) \oplus (\mathcal{S}^- \otimes \mathcal{S}^{*-})) \bigoplus ((\mathcal{S}^+ \otimes \mathcal{S}^{*-}) \oplus (\mathcal{S}^- \otimes \mathcal{S}^{*+})).$$

$Cl(T^*M)$  also has an action on it. Under the above isomorphism (1.4), these two connections, graded structure, action are identical (see Berline-Getzler-Vergne [6] Chapter 3 and 4). So from now we will always consider exterior form bundle  $\Omega^*(M)$  as twisted Clifford module bundle  $\mathcal{S} \otimes \mathcal{S}^*$ .

**Lemma 2.1.** *Let  $T$  denote Berezin integral (see Berline-Getzler-Vergne [6]). For  $a \in \text{End}(\mathcal{S}_x) \cong Cl(T_x^*M)$ ,  $b \in \text{End}(\mathcal{S}_x^*) \cong Cl(T_x^*M)$ ,*

$$\text{End}(\mathcal{S}_x) \otimes \text{End}(\mathcal{S}_x^*) \cong \text{End}(\wedge^*(T_x^*M \otimes_R C)),$$

$$\text{Str}(a) = \text{tr}(\Gamma a) = (-2i)^l T \circ \sigma(a),$$

$$\text{Str}(b) = \text{tr}(\Gamma^* b) = (2i)^l T \circ \sigma(b),$$

$$\text{Str}(a \otimes b) = \text{tr}(\Gamma a) \text{tr}(\Gamma^* b).$$

For the Dirac operator  $\mathcal{D}^{\mathcal{S}}$  associated with  $D^{\mathcal{S}}$ :

$$\mathcal{D}^{\mathcal{S}} = c(e^i) D_{e_i}^{\mathcal{S}},$$

there exist unique 1-form  $a = a_i e^i$ , 3-form  $B = B_{il} e^i \wedge e^l \wedge e^s$ , such that

$$\mathcal{D}^{\mathcal{S}} = \widehat{\mathcal{D}}^{\mathcal{S}} + c(a) + c(B).$$

Define

$$D_{e_i}^{\mathcal{S}, B} = \widehat{D}_{e_i}^{\mathcal{S}} + B_{il}^s c(e^l) c(e^s),$$

then  $\mathcal{D}^{\mathcal{S}, B} = \widehat{\mathcal{D}}^{\mathcal{S}} + c(B)$ .

Let  $W$  be a complex bundle equipped with connection  $D^W$ , curvature be  $F$ . Define the connections  $\widehat{D}^{\mathcal{S} \otimes W}, D^{\mathcal{S} \otimes W}, D^{\mathcal{S} \otimes W, B}, D^{\mathcal{S} \otimes W, 3B}$ , whose corresponding Dirac operator are  $\widehat{\mathcal{D}}^{\mathcal{S} \otimes W}, \mathcal{D}^{\mathcal{S} \otimes W}, \mathcal{D}^{\mathcal{S} \otimes W, B}, \mathcal{D}^{\mathcal{S} \otimes W, 3B}$ :

$$\widehat{D}^{\mathcal{S} \otimes W} = \widehat{D}^{\mathcal{S}} \otimes 1 + 1 \otimes D^W,$$

$$D^{\mathcal{S} \otimes W} = D^{\mathcal{S}} \otimes 1 + 1 \otimes D^W,$$

$$\begin{aligned} D^{\mathcal{S} \otimes W, B} &= D^{\mathcal{S}, B} \otimes 1 + 1 \otimes D^W, \\ D^{\mathcal{S} \otimes W, 3B} &= D^{\mathcal{S}, 3B} \otimes 1 + 1 \otimes D^W. \end{aligned}$$

A useful formula on the square of  $\mathcal{D}$  is from Bismut [1].

**Lemma 2.2.** (*Bismut [1]*)

$$(\mathcal{D}^{\mathcal{S} \otimes W, B})^2 = -\Delta^{\mathcal{S} \otimes W, 3B} + \frac{s}{4} + c(F) + c(dB) - 2|B|^2.$$

$$\begin{aligned} (\mathcal{D}^{\mathcal{S} \otimes W})^2 &= (\mathcal{D}^{\mathcal{S} \otimes W, B})^2 - 2(a, e^i) \hat{D}_{e_i}^{\mathcal{S} \otimes W} + c(\hat{D}a) - 2c(\iota(a)B) - |a|^2 \\ &= -\Delta^{\mathcal{S} \otimes W, 3B} - 2(a, e^i) D_{e_i}^{\mathcal{S} \otimes W, 3B} + C, \end{aligned}$$

where  $C = \frac{s}{4} + c(F) + c(dB) - 2|B|^2 + c(\hat{D}a) - |a|^2$ ,  $s$  is scalar curvature.

Define the connections  $\hat{D}^{\mathcal{S} \otimes \mathcal{S}^*}$ ,  $D^{\mathcal{S} \otimes \mathcal{S}^*}$ ,  $D^{\mathcal{S} \otimes \mathcal{S}^*, B}$ ,  $D^{\mathcal{S} \otimes \mathcal{S}^*, 3B}$  on  $\mathcal{S} \otimes \mathcal{S}^*$ , and corresponding Dirac operators are noted by  $\hat{\mathcal{D}}^{\mathcal{S} \otimes \mathcal{S}^*}$ ,  $\mathcal{D}^{\mathcal{S} \otimes \mathcal{S}^*}$ ,  $\mathcal{D}^{\mathcal{S} \otimes \mathcal{S}^*, B}$ ,  $\mathcal{D}^{\mathcal{S} \otimes \mathcal{S}^*, 3B}$ .

$$\begin{aligned} \hat{D}^{\mathcal{S} \otimes \mathcal{S}^*} &= \hat{D}^{\mathcal{S}} \otimes 1 + 1 \otimes D^{\mathcal{S}^*}, \\ D^{\mathcal{S} \otimes \mathcal{S}^*} &= D^{\mathcal{S}} \otimes 1 + 1 \otimes D^{\mathcal{S}^*}, \\ D^{\mathcal{S} \otimes \mathcal{S}^*, B} &= D^{\mathcal{S}, B} \otimes 1 + 1 \otimes D^{\mathcal{S}^*}, \\ D^{\mathcal{S} \otimes \mathcal{S}^*, 3B} &= D^{\mathcal{S}, 3B} \otimes 1 + 1 \otimes D^{\mathcal{S}^*}. \end{aligned}$$

When consider  $\Omega^*(M)$  as  $\mathcal{S} \otimes \mathcal{S}^*$ ,  $D^{\mathcal{S} \otimes \mathcal{S}^*}$  and  $\mathcal{D}^{\mathcal{S} \otimes \mathcal{S}^*}$  are respectively  $D$  and  $\mathcal{D}$  defined in theorem 1.1. So we can get the expression of  $\mathcal{D}^2$  by lemma 2.2. This is the key step.

### 3. THE PROOF OF THE MAIN THEOREM

In the following proof, we use Feynman-Kac formula and the generalized Wiener functional which are included in stochastic analysis. In the course of studying Malliavin theory, the generalized Wiener functional and its applications were introduced and studied by Malliavin, Kusuoka-Stroock [14], Watanabe [11] [12], Ikeda-Watanabe [13] and so on. In this paper we adopt the definition and processing mode as in Watanabe [12]. Watanabe [12] proved the local Gauss-Bonnet-Chern theorem and signature theorem by this method. About more details on generalized Wiener functional and its applications, the readers could refer to Ikeda-Watanabe [13]. The probabilistic proof on index theorem was provided firstly by Bismut [2]. Base on probabilistic method, Bismut [1] proved a local index theorem on non Kähler manifold. There was still a stochastic proof for the local Gauss-Bonnet-Chern theorem in Elton Hsu [19]. If not using stochastic method, the main theorem in this article should be also able to be proved by Getzler's rescaling method as in Berline-Getzler-Vergne [6] and Getzler [20]. The most notations in our computations are the same as Watanabe [12].

Furthermore, we assume  $M$  be  $R^d$ , with metric  $g$ , which is equal to the standard Euclidean metric outside of some sufficient big ball, the

natural coordinate on  $R^d$  be identical to normal coordinate around the original point .

Let  $\{e_i\}$  be natural tangent frame,  $\{f_i\}, \{g_i\}$  be orthonormal frames respectively by parallel translations along the radial lines from original point under connections  $D^{3B}, D$  on  $M$ ,  $\{e^i\}, \{f^i\}, \{g^i\}$  are respectively their dual frames. Let connection  $D^{3B} = (\Gamma^{3B})_{il}^s e^i \otimes f_s \otimes f^l$ , under frame  $\{f_i\}$  ; connection  $D = \Gamma_{il}^s e^i \otimes g_s \otimes g^l$  under frame  $\{g_i\}$ . By virtue of the above frames, we could get the trivialization of  $\mathcal{S} \otimes \mathcal{S}^*$ :  $R^d \times (S \otimes S^*)$ , where  $S$  is the spinor space of Euclidean space  $R^d$ .

Let  $C_i = \frac{1}{4}(\Gamma^{3B})_{il}^s c(f^l)c(f^s) \otimes 1 + 1 \otimes (-\frac{1}{4}\Gamma_{il}^s) c^*(g^s)c^*(g^l)$ ,  $b^i = -g^{ls}\hat{\Gamma}_{ls}^i + 2(a, e^i)$ , then according to lemma 2.2, the Hodge-Laplacian heat equation expression in the natural coordinate is:

$$\begin{cases} \frac{\partial f}{\partial t} = \frac{1}{2}g^{ij}(\frac{\partial}{\partial x^i} + C_i)(\frac{\partial}{\partial x^j} + C_j)f + \frac{1}{2}b^i(\frac{\partial}{\partial x^i} + C_i)f - \frac{1}{2}Cf, & (t, x) \in (0, \infty) \times R^d \\ f(0, x) = \varphi(x), & x \in R^d \end{cases}$$

Let smooth real symmetric positive matrix  $\sigma_k^i$  make

$$\sum_k \sigma_k^i \sigma_k^l = g^{il},$$

then consider the following stochastic differential equation valued in  $R^d \times (Cl(R^d) \otimes Cl(R^d)) \times (Cl(R^d) \otimes Cl(R^d))$ ,

$$\begin{cases} dX^i(t) = \sigma_k^i(X(t))dw^k(t) + \frac{1}{2}b^i(X(t))dt, \\ de(t) = e(t)C_i(X(t)) \circ dX^i(t), \\ dM(t) = -\frac{1}{2}M(t)e(t)C(X(t))e^{-1}(t)dt, \\ e(0) = 1, M(0) = 1, X(0) = x, \end{cases}$$

$e(t)$  are inverse almost everywhere (see Stroock [18]), so  $e^{-1}(t)$  are well defined.

By Itô formula and properties of generalized Wiener functional ,

$$f(t, x) = E[M(t)e(t)\varphi(X_x(t))],$$

$$h(t, x, y) = E[M(t)e(t)\delta_y(X(t))],$$

where  $\delta_y$  is a generalized function: the Dirac delta function associated with  $y$ . It is difficult that compute directly asymptotic expansion for  $t$  according to the above formula. As in Bismut [2], Watanabe [12], we consider the stochastic differential equations with parameter  $\varepsilon$ :

$$\begin{cases} dX^i(t) = \varepsilon \sigma_k^i(X(t))dw^k(t) - \frac{\varepsilon^2}{2}b^i(X(t))dt, \\ de(t) = e(t)C_i(X(t)) \circ dX^i(t), & i, j, k = 1, 2, \dots, d. \\ (X(0), e(0)) = (0, 1), \\ \begin{cases} dM(t) = -\frac{\varepsilon^2}{2}M(t)e(t)C(X(t))e^{-1}(t)dt, \\ M(0) = 1, \end{cases} \end{cases}$$

let us denote the solution by  $r^\varepsilon(t) = (X^\varepsilon(t), e^\varepsilon(t), M^\varepsilon(t))$ , then

$$h(\varepsilon^2, 0, 0) = E[M^\varepsilon(1)e^\varepsilon(1)\delta_0(X^\varepsilon(1))].$$

**Lemma 3.1.** (see Berline-Getzler-Vergne [6])

$$\Gamma_i(x) = -\frac{1}{2} \sum_j R(\partial_i, \partial_j)(0)x^j + O(|x|^2).$$

$$\Gamma_i^{3B}(x) = -\frac{1}{2} \sum_j R^{3B}(\partial_i, \partial_j)(0)x^j + O(|x|^2).$$

**Lemma 3.2.** (see Watanabe [11] [12]) Let  $D^\infty$  be the space consists of  $R^d$  valued Wiener functionals whose any order Malliavin derivatives are  $L_p$  integrable, for all  $p > 1$ ,  $\tilde{D}^{-\infty}$  be its dual space.

$$X^\varepsilon(1) = \varepsilon w(1) + O(\varepsilon^2)$$

in  $D^\infty$ .

$$\delta_0(X^\varepsilon(1)) = \varepsilon^{-d}\delta_0(w(1)) + O(\varepsilon^{-d+1})$$

in  $\tilde{D}^{-\infty}$ .

$$E[\delta_0(w(1)) \cdot \Phi(w)] = (2\pi)^{-l} E[\Phi(w)|w(1) = 0], \quad \Phi \in \tilde{D}^\infty.$$

Let

$$\begin{aligned} \theta^\varepsilon(t) &= \int_0^t C_i(X^\varepsilon(t)) \circ d(X^\varepsilon)^i(s) \\ &= \varepsilon^2(C_{ij}^1(t)c(f^i)c(f^j) \otimes 1 + 1 \otimes C_{ij}^2(t)c^*(g^i)c^*(g^j)) + O(\varepsilon^3), \end{aligned}$$

in which

$$C_{ij}^1(t) = \frac{1}{8} R_{mki j}^{3B}(0) \int_0^t w^k(s) \circ dw^m(s),$$

$$C_{ij}^2(t) = \frac{1}{8} R_{mki j}(0) \int_0^t w^k(s) \circ dw^m(s),$$

$$R_{mki j} = (R(e_m, e_k)g_j, g_i), R_{mki j}^{3B} = (R^{3B}(e_m, e_k)f_j, f_i).$$

let

$$B^\varepsilon(t) = \theta^\varepsilon(t) - \int_0^t \frac{\varepsilon^2}{2} C(X^\varepsilon(s)) ds.$$

$$\begin{aligned}
M^\varepsilon(1)e^\varepsilon(1) &= 1 + \int_0^1 M^\varepsilon(s)e^\varepsilon(s) \circ d\theta^\varepsilon(s) + \int_0^1 M^\varepsilon(s)e^\varepsilon(s) \left(-\frac{\varepsilon^2}{2}C(X^\varepsilon(s))\right) ds \\
&= 1 + B^\varepsilon(1) + \int_0^1 \int_0^{t_1} M^\varepsilon(t_2)e^\varepsilon(t_2) \circ dB^\varepsilon(t_2) \circ dB^\varepsilon(t_1) \\
&= 1 + B^\varepsilon(1) + \int_0^1 B^\varepsilon(t_1) \circ dB^\varepsilon(t_1) \\
&\quad + \int_0^1 \int_0^{t_1} \int_0^{t_2} M^\varepsilon(t_3)e^\varepsilon(t_3) \circ dB^\varepsilon(t_3) \circ dB^\varepsilon(t_2) \circ dB^\varepsilon(t_1) \\
&= 1 + A_1 + A_2 + \dots + A_l + O(\varepsilon^{2l+2})
\end{aligned}$$

in  $D^\infty(Cl(R^d) \otimes End(R^s))$ ,

in which

$$\begin{aligned}
A_m &= \int_0^1 \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_{m-1}} \circ dB^\varepsilon(t_m) \circ dB^\varepsilon(t_{m-1}) \circ \dots \circ dB^\varepsilon(t_1) \\
&= \varepsilon^{2m} \int_0^1 \int_0^{t_1} \dots \int_0^{t_{m-1}} \circ d\tilde{C}(t_m) \circ d\tilde{C}(t_{m-1}) \circ \dots \circ d\tilde{C}(t_1) + O(\varepsilon^{2m+1})
\end{aligned}$$

in  $D^\infty(Cl(R^d) \otimes End(R^s))$ ,

$$\tilde{C}(t) = C_{ij}^1(t)c(f^i)c(f^j) \otimes 1 + 1 \otimes C_{ij}^2(t)c^*(g^i)c^*(g^j) - \int_0^t \frac{1}{2}C(0)dt.$$

Note lemma 2.1, when  $m < l$ ,

$$Str(A_m) = 0,$$

$$m = l,$$

$$Str(A_m) = Str(A_l),$$

$$m > l,$$

$$Str(A_m) = O(\varepsilon^{2l+3}) = O(\varepsilon^{d+3}).$$

$$\begin{aligned}
Str(A_l) &= \frac{\varepsilon^{2l}}{l!} Str\left\{\left(-\frac{1}{2}C(0)\right)^l\right\} + O(\varepsilon^{2l+1}) \\
&= \frac{\varepsilon^{2l}}{l!} Str\left\{\left(-\frac{1}{2}c(F)(0)\right)^l\right\} + O(\varepsilon^{2l+1}) \\
&= \frac{\varepsilon^{2l}}{l!} Str\left\{\left(-\frac{1}{4}\left(\frac{1}{4}R_{ijnm}(0)c(g^i)c(g^j)c^*(g^m)c^*(g^n)\right)\right)^l\right\} + O(\varepsilon^{2l+1}) \\
&= \varepsilon^{2l}(-2i)^{2l}(-1)^l Pf\left(-\frac{1}{4}R(0)\right) + O(\varepsilon^{2l+1}) \\
&= \varepsilon^{2l}Pf(-R(0)) + O(\varepsilon^{2l+1})
\end{aligned}$$

Note lemma 3.2, when  $\varepsilon \rightarrow 0$

$$Str[h(\varepsilon^2, 0, 0)]e^1 \wedge e^2 \wedge \dots \wedge e^d = \frac{1}{(2\pi)^l} Pf(-R(0)) + O(\varepsilon).$$

Therefore, we get

$$\lim_{t \rightarrow 0} \text{Str}[h(t, 0, 0)]e^1 \wedge e^2 \wedge \dots \wedge e^d = \frac{1}{(2\pi)^l} Pf(-R(0)).$$

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